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# Resistance Effects on Hydraulic Instability

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For almost two centuries many quantitative experiments have been made concerning the flow of water down an inclined open channel. Hydraulicians have evolved empirical laws in confusing multiplicity to describe a uniform flow as a function of the slope and the resistance caused by friction with the channel walls and by turbulence. Each of the laws has its own best range of applicability, but most are of the same basic type, viz. the effective resistance is considered in the momentum equation as an opposing body force of form $-\lambda u^n/y^m$, $u$ being the velocity, $y$ the "hydraulic radius" and $\lambda$ a coefficient describing the channel roughness, which in this notation is independent of $u$ and $y$. The exponents $m, n > 0$ have specific values characteristic of the different formulas. For a wide rectangular channel, $y$ will be also the depth of the water.

A uniform (turbulent) flow with a free surface, being subject to some such resistance law, sometimes reaches a condition of instability, then changes into a complicated progressing wave pattern called "roll waves" which are periodic in distance, with the profile moving downstream faster than the water particles. Mathematical analyses have been made of certain aspects of the final roll wave motion by Thomas [1] and one of the present authors [2]. Results giving necessary criteria have been obtained by considering the correct slope of the roll wave profile or by satisfying the energy inequality at the moving discontinuities (bores). Our present purpose, however, is rather to investigate the instability of the original uniform flow in its dependence upon the specific resistance function assumed, following the method of Jeffreys [3] who first did this in 1925 for the Chezy formula ($n = 2$, $m = 1$ in our present notation). The criterion obtained by Jeffreys for instability is identical with the condition later obtained for roll wave formation in [1] and [2]. Each of the papers [1], [2] and [3] used the Chezy resistance function.

In 1940 Keulegan and Patterson [4] derived a stability criterion for the Manning formula, based upon an expression for wave celerity due to Boussinesq. Another type of stability argument was later used by Vedernikov [5], using certain approximations of Saint Venant. His method is rather complicated, but the results are applicable for a general resistance term. More recently Craya [6] has refined this approach by considering the growth of an infinitesimal shock ("elementary wave") with results applying also in the case of a general resistance. From the qualitative standpoint, Cornish [7], with appendix by Harold Jeffreys, has presented some interesting remarks on the formation of roll waves with respect to the resistive action of the stream bed.

Using the shock energy inequality to construct actual roll wave solutions, it was observed in [2] that such solutions could never be constructed if the variation of resistance effect with depth was ignored. That is, if the resistance depended only on the square of the velocity, the energy shock condition for the bores of the waves would always be violated. Such a simplifying assumption on the resistance is sometimes made, however, particularly when variations in depth are considered to be relatively small. We wish now to determine whether agreement between the stability approach and the roll wave approach extends also to this case, and to see more generally how instability depends on the form of the resistance function. That is, one wishes to see what particular quantities in the resistance mechanism can lead to instability and roll waves. This present investigation will adapt the method of Jeffreys to a general resistance function, and results will be compared with those obtained by the other authors mentioned above, using different approaches to the problem.

In a two-dimensional flow, let $u$ be the velocity component parallel to the stream bed, inclined at an angle $\theta$ below the horizontal, and let $y$ be the depth of water. The non-linear shallow water equations are then

$$(1)\qquad \begin{aligned} u_t + uu_x + g\cos\theta y_x &= g\sin\theta - (\lambda u^n/y^m) \\ yu_x + uy_x + y_t &= 0. \end{aligned}$$

The Chezy formula corresponds to $m = 1$, $n = 2$; the Manning formula to $m = 4/3$, $n = 2$; while in the various Lea and Barnes expressions, $m$ varies from 1.13 to 1.64 and $n$ from 1.7 to 2.15 (see King [8]).

The possible uniform flows are given by $u = U$, $y = Y$ where

$$(2)\qquad U = \left(\frac{gY^m \sin\theta}{\lambda}\right)^{1/n}.$$

Considering small deviations given by $u = U + \bar{u}(x,t)$, $y = Y + \bar{y}(x,t)$ and linearizing yield two equations for $\bar{u}$ and $\bar{y}$,

$$(3)\qquad \begin{aligned} \bar{u}_t + U\bar{u}_x + g\cos\theta\bar{y}_x &= \frac{gm\sin\theta}{Y}\bar{y} - \lambda\frac{nU^{n-1}}{Y^m}\bar{u} \\ U\bar{y}_x + Y\bar{u}_x + \bar{y}_t &= 0. \end{aligned}$$

When $\bar{y}$ is eliminated, we get the second order linear equation for $\bar{u}(x,t)$,

$$(4)\qquad \begin{aligned} \bar{u}_{tt} + 2U\bar{u}_{xt} + (U^2 - gY\cos\theta)\bar{u}_{xx} + \frac{\lambda nU^{n-1}}{Y^m}\bar{u}_t & \\ + \left(mg\sin\theta + \frac{\lambda nU^{n-1}}{Y^m}\right)\bar{u}_x &= 0. \end{aligned}$$

We now study the time growth or decay of the class of progressing wave solu-

tions of type

$$u = Ae^{\gamma t}\binom{\cos}{\sin}\left(\beta\left[x + \frac{s}{\beta}t\right]\right)$$

with wave speed $s/\beta$ and wave length proportional to $1/\beta$. Let $\gamma = r + is$; then insertion of the solutions $A \exp\{\gamma t + i\beta x\}$ into (4) implies the following relation between $\gamma$ and $\beta$

$$\gamma = -\frac{\lambda n U^{n-1}}{2Y^m} - iU\beta \pm \left(\frac{\lambda^2 n^2 U^{2n-2}}{4Y^{2m}} - \beta^2 gY \cos\theta - i\beta gm \sin\theta\right)^{1/2}. \tag{5}$$

For stability we require both roots to be in the left half of the complex $\gamma$-plane. If parameters are adjusted so that the real part of the first square root in (5) is equal to $\lambda n U^{n-1}/2Y^m$, then the corresponding $\gamma$ value will be on the imaginary axis, and the other value in the left half-plane. We therefore equate this square root to $\lambda n U^{n-1}/2Y^m + i\delta$, separate reals and imaginaries, eliminate $\delta$, and use relation (2). The condition for the right root to be on the $s$-axis is then

$$m^{2n} g^{2-n} Y^{2m-n} \sin^2\theta = n^{2n}\lambda^2 \cos^n\theta. \tag{6}$$

When the expression on the left is greater or smaller, it can be shown that the root moves to the left or right, respectively. Considering the expressions in (6) as functions of $\theta$, we see that equation (6) must have exactly one $\theta$ solution between 0 and $\pi/2$. The stability criterion for general resistance is thus

$$m^{2n} g^{2-n} Y^{2m-n} \sin^2\theta \lesseqgtr n^{2n}\lambda^2 \cos^n\theta \tag{7}$$

implying stability for the two top signs, and instability for the bottom sign. The condition for instability can be put into an equivalent form, using (2),

$$U > \frac{n}{m}\sqrt{gY\cos\theta}, \tag{8}$$

which is identical with the Vedernikov and Craya results, obtained by other methods and assumptions.

The above instability criterion for the Chezy formula reduces to $\tan\theta > 4\lambda$, the result obtained in [1], [2], and [3]. For the Manning formula ($m = 4/3$, $n = 2$), our result is

$$U^2 > \frac{9}{4}gY\cos\theta \tag{9}$$

which agrees with the result of Keulegan and Patterson except for the absence of the factor $\cos\theta$ in their criterion.

To see what is the essential non-linearity in (1) causing instability, we now consider two special limit cases of (7):

*Case* 1: $m \to 0, n > 0$.

Now (7) reduces to only one possibility

$$0 < n^{2n} \lambda^2 \cos^n \theta,$$

hence we conclude that any flow governed by a resistance varying with any power of velocity, but independent of depth, *must remain stable*.

*Case* 2: $n \to 0, m > 0$.

By virtue of the law for uniform regime $Y = (\lambda / g \sin \theta)^{1/m}$, the only possibility for (7) is the equality

$$g^2 Y^{2m} \sin^2 \theta = \lambda^2.$$

Both $\gamma$ roots move to the $x$-axis, *still a region of stability*. Although this case, describing a type of hydraulic Coulomb friction, may not correspond to actual flows, the analysis nevertheless indicates that *both* effects of depth and velocity dependence operating simultaneously are needed to produce instability. For this reason, since one-dimensional compressible gas flow is governed by the same equations, with resistance depending only on velocity, one would not expect any analogous type of instability to develop in such flows.

## Bibliography


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